

Ward-Takahashi identity with an external field in ladder QED

Efrain J. Ferrer and Vivian de la Incera

Physics Department, State University of New York, Fredonia, New York 14063

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We derive the Ward-Takahashi identity obeyed by the fermion-antifermion-gauge boson vertex in ladder QED in the presence of a constant magnetic field. The general structure in momentum space of the fermion mass operator with an external electromagnetic field is discussed. Using it we find solutions of the ladder WT identity with a magnetic field. The consistency of our results with the solutions of the corresponding Schwinger-Dyson equation ensures the gauge invariance of the magnetic-field-induced chiral symmetry breaking recently found in ladder QED. [S0556-2821(98)04116-2]

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I. INTRODUCTION

Recently, the problem of finding a mechanism to generate galactic magnetic fields has undergone a boost of attention [1–3]. The interest has been motivated by the experimental observation in our own, as well as in other galaxies, of magnetic fields of about 10^{-6} G on scales of the order of 100 kpc. The connection between the direction of the observed galactic magnetic fields and the rotation axis of the galaxy suggests that they have been produced by a dynamo effect, in which a small seed magnetic field is amplified by the turbulent motion of matter during galaxy formation [4]. This fact leads to the problem of the generation of seed primordial fields, which has been addressed in many papers [1–3], where a large number of different generating mechanisms have been proposed.

Parallel to these investigations, and perhaps partially motivated by them, several fundamental studies on the effects produced by magnetic fields in quantum field theories [5–16] have been recently carried out. We would like to refer to a particularly interesting effect of a magnetic background, known as the catalysis of chiral symmetry breaking due to a magnetic (chromomagnetic, hypermagnetic) field. The essence of this effect lies in the dimensional reduction in the dynamics of fermion pairing in the presence of a magnetic field [6]. Because of such a dimensional reduction, the magnetic field catalyzes the generation of a fermion condensate and, consequently, of a dynamical fermion mass, even in the weakest attractive interaction between fermions.

An important aspect of magnetic-field-induced chiral symmetry breaking (MCSB) is related to its possible cosmological consequences. It has been speculated [6] that the character of electroweak phase transitions could be affected by MCSB. The existence of a broken chiral symmetry phase due to a magnetic field in the early universe would require the presence of very large primordial magnetic fields at those stages.

As mentioned above, the existence of primordial magnetic fields in the early universe cannot be disregarded, but on the contrary, it seems to be needed to explain the observed large-scale galactic magnetic fields. Several primordial-field-generating mechanisms predict fields as large as 10^{24} G during the electroweak phase transitions. Moreover, Ambjørn and Olesen [17] have claimed that seed fields even larger, $\sim 10^{33}$ G, would be necessary at the electroweak phase tran-

sition to explain the observed galactic fields. Notwithstanding the possible existence of such large seed fields, it has been argued [8,13] that they are too small to make MCSB an important effect during the electroweak phase transition. This conclusion was drawn from the study of MCSB in QED at finite temperature. Nevertheless, in recent calculations of the catalysis of chiral symmetry breaking in an Abelian gauge model with Yukawa terms (which is a model with many of the attributes of the fermion sector of the electroweak theory), it has been found [18] that the critical magnetic field, required for this effect to be important at the electroweak scale, can be substantially decreased by a Yukawa coupling of the order of the one corresponding to the top quark. From our point of view, these results indicate that the problem of the influence of MCSB in the electroweak transition is still open and deserves more investigation.

The catalysis of chiral symmetry breaking by a magnetic field has been found in a variety of theories such as Nambu–Jona-Lasinio [6,10,15], QED [6–11], and QCD [12,16]. These studies have been done using a Bethe-Salpeter (BS) [6] or a Schwinger-Dyson (SD) approach [7–9], at zero [6,7] and at finite [8,13] temperature, as well as in QED at finite density [9] (i.e., in the presence of a chemical potential). In all cases the dynamical mass was the result of a nonperturbative calculation defined by the ladder and/or improved ladder approximations.

We must recall, however, that the so-called ladder or rainbow approximation is not gauge invariant [19]. One may wonder, then, whether the magnetic-field-induced chiral mass is for real or just an artifact of the approximation used.

Indeed this question is not new. In the absence of any external field, a similar problem [20] is present in the strong coupling regime of QED, where the critical coupling for chiral symmetry breaking, found by solving the SD (or the Bethe-Salpeter) equation for the fermion propagator in the ladder approximation, is gauge dependent [19]. The solution to this problem was obtained by investigating the Ward-Takahashi (WT) identities in the ladder approximation. As known, if the WT identities are satisfied by solution of the SD equations in some approximation in a certain gauge, one can use the gauge transformation law for the Green's functions [21] to rewrite the SD equations in an arbitrary gauge. The transformation law guarantees that the WT identities are satisfied by solutions of the SD equations in all other gauges,

although the approximation on which the SD equations are solved may change. As it turns out, the gauge invariance of the critical coupling found in ladder QED [19] is ensured by the consistency of the WT identity with the SD solution, taken in the same ladder approximation in the Landau gauge.

When a magnetic field is present, one can follow a similar strategy to prove the gauge invariance of the magnetic-field-induced chiral mass obtained in ladder QED. This program is far from trivial though. On the one hand, the fermion Green's function cannot be diagonalized in momentum coordinates by a Fourier transform. To diagonalize it, one must work in the representation of the eigenfunctions of the fermion mass operator [22,23]. On the other hand, the magnetic field provides the theory with an extra tensor $F_{\mu\nu}$, making the structure of the fermion mass operator much richer [22,24]. The new structural terms modify the WT identities of the theory, which now involve a larger number of scalar coefficients (the scalar coefficients of the different independent structures of the mass operator).

The main goal of the present paper is to establish the gauge invariance of the magnetic-field-induced chiral mass found in ladder QED, taking into account the above-mentioned features of the theory in the presence of a magnetic field. With this aim, we study the ladder WT identity for the fermion-antifermion-gauge boson vertex in the presence of a magnetic field. Recalling that the consistency of the WT and SD solutions should be established within the same approximation, we solve the ladder WT identity in the lowest Landau level (LLL). As shown below, the solutions of the WT and SD equations are indeed compatible in the Feynman gauge within this approximation.

The plan of the paper is the following. In Sec. I, we derive the momentum representation of the WT identity for ladder QED in a magnetic field. To solve the identity, we need to use the fermion mass propagator, whose structure in the p -space is discussed in Sec. II. The WT solution is finally found in Sec. III for fermions in the LLL. As we discuss there, it is satisfied by the solution, obtained in Refs. [7,8], of the corresponding SD equation within the same approximation. Hence, the gauge invariance of the chiral mass is proved. In Sec. IV we give the conclusions.

II. WARD-TAKAHASHI IDENTITY IN LADDER QED WITH AN EXTERNAL MAGNETIC FIELD

Let us derive the WT identity in momentum space for the fermion-antifermion-gauge boson vertex in QED in the presence of a constant magnetic field. In coordinate representation the WT identity looks the same as in the case without a magnetic field:

$$\begin{aligned} \partial_\mu^z \frac{\delta^3 \Gamma(0)}{\delta \bar{\Psi}(x) \delta \Psi(y) \delta A_\mu(z)} &= ie \delta^{(4)}(y-z) \frac{\delta^2 \Gamma(0)}{\delta \bar{\Psi}(x) \delta \Psi(y)} \\ &\quad - ie \delta^{(4)}(x-z) \frac{\delta^2 \Gamma(0)}{\delta \bar{\Psi}(x) \delta \Psi(y)}, \end{aligned} \quad (1)$$

where Γ is the generating functional for proper vertices, and the zero in Γ means that all average fields, but the magnetic field, are set to zero. The existence of the external magnetic field is manifested in the modification of the fermion propagator as discussed below.

The solutions of Eq. (1) depend on the approximation used for the vertex and the inverse propagator. As discussed in the Introduction, we are interested in the study of the identity (1) in the same approximation used in Refs. [7,8] to solve the SD equation for the fermion propagator. Therefore, we need to use the so-called rainbow or ladder approximation, in which the vertex is taken as bare,

$$\frac{\delta^3 \Gamma(0)}{\delta \bar{\Psi}(x) \delta \Psi(y) \delta A^\mu(z)} = -e \delta^{(4)}(x-z) \delta^{(4)}(y-z) \gamma_\mu, \quad (2)$$

and the fermion inverse propagator is taken as full (denoted by a bar over S^{-1}),

$$\frac{\delta^2 \Gamma(0)}{\delta \bar{\Psi}(x) \delta \Psi(y)} = \bar{S}^{-1}(x, y). \quad (3)$$

In this nonperturbative approximation the WT identity (1) becomes

$$\begin{aligned} -\partial_z^\mu [\delta^{(4)}(x-z) \delta^{(4)}(y-z) \gamma_\mu] \\ = i \delta^{(4)}(y-z) \bar{S}^{-1}(x, y) - i \delta^{(4)}(x-z) \bar{S}^{-1}(x, y). \end{aligned} \quad (4)$$

We are interested in the momentum representation of this identity. Usually, the transformation to a momentum representation can be done by Fourier transforming Eq. (1). In the presence of an external background field, however, the Fourier transform of the fermion Green's functions is not diagonal in p space. Fortunately, in the case of a constant uniform magnetic field H , the fermion propagator, and hence its inverse, can be p -diagonalized by working in the so-called E_p representation [23].

The E_p representation, introduced by Ritus [23] in his studies of QED with external electromagnetic fields, is based on the eigenfunctions of the operator $(\gamma \cdot \Pi)^2$:

$$(\gamma \cdot \Pi)^2 \psi_p = \bar{p}^2 \psi_p. \quad (5)$$

The operator $(\gamma \cdot \Pi)^2$ commutes with the mass operator. Hence, it is convenient to work with the mass operator expressed in the representation defined by these eigenfunctions.

In the chiral representation in which γ_5 and $\Sigma_3 = i \gamma^1 \gamma^2$ are diagonal with eigenvalues ± 1 , the eigenfunctions ψ_p are given by

$$\psi_p = E_{p\sigma}(x) \omega_\sigma = N(n) e^{i(p_0 x^0 + p_2 x^2 + p_3 x^3)} D_n(\rho) \omega_\sigma, \quad (6)$$

where ω_σ are the proper bispinors of γ_5 and Σ_3 , $D_n(\rho)$ are parabolic cylinder functions [25] with argument $\rho = \sqrt{2|eH|}(x_1 - p_2/eH)$ and positive integer index:

$$n = n(k, \sigma) \equiv k + \frac{eH\sigma}{2|eH|} - \frac{1}{2}, \quad n = 0, 1, 2, \dots \quad (7)$$

$N(n) = (4\pi|eH|)^{1/4}/\sqrt{n!}$ is a normalization factor. Here p represents the set (p_0, p_2, p_3, k) , which determines the eigenvalue $\bar{p}^2 = -p_0^2 + p_3^2 + 2|eH|k$ in Eq. (5).

The E_p representation is obtained forming the eigenfunction matrices (see Refs. [7,8,23] for details)

$$E_p(x) = \sum_{\sigma} E_{p\sigma}(x) \Delta(\sigma), \quad (8)$$

where

$$\Delta(\sigma) = \text{diag}(\delta_{\sigma 1}, \delta_{\sigma -1}, \delta_{\sigma 1}, \delta_{\sigma -1}), \quad \sigma = \pm 1. \quad (9)$$

It is easy to check that the E_p functions are orthonormal,

$$\begin{aligned} & \int d^4x \bar{E}_{p'}(x) E_p(x) \\ &= (2\pi)^4 \delta^{(4)}(p - p') \\ &\equiv (2\pi)^4 \delta_{kk'} \delta(p_0 - p'_0) \delta(p_2 - p'_2) \delta(p_3 - p'_3), \end{aligned} \quad (10)$$

as well as complete:

$$\sum_k \int dp_0 dp_2 dp_3 E_p(x) \bar{E}_p(y) = (2\pi)^4 \delta^{(4)}(x - y). \quad (11)$$

Here we used $\bar{E}_p(x) = \gamma^0 E_p^\dagger \gamma^0$.

They also satisfy two important relations

$$\gamma \cdot \Pi E_p(x) = E_p(x) \gamma \cdot \bar{p}, \quad (12)$$

$$\int d^4x' M(x, x') E_p(x) = E_p(x) \tilde{\Sigma}_A(\bar{p}). \quad (13)$$

$M(x, x')$ is the matrix element of the mass operator in coordinate representation. Equation (13) is then the eigenvalue equation for the mass operator, with $\tilde{\Sigma}_A(\bar{p})$ being the eigenvalue matrix of the mass operator in momentum space. Notice that the eigenvalue matrix $\tilde{\Sigma}_A(\bar{p})$ is evaluated in $\bar{p} = (p_0, 0, -\text{sgn}(eH)\sqrt{2|eH|k_p}, p_3)$.

Let us transform Eq. (4) to momentum space. First, note that in Eq. (4) the fermion inverse propagators do not depend on z . Therefore, the transformation to momentum coordinates for the variable z can be done with the usual Fourier transform. Multiplying Eq. (4) by e^{iqz} and integrating in z one finds

$$\delta^{(4)}(x - y) e^{iqx} q^\mu \gamma_\mu = e^{iqy} \bar{S}^{-1}(x, y) - e^{iqx} \bar{S}^{-1}(x, y). \quad (14)$$

If one now multiplies Eq. (14) by the left by $\bar{E}_p(x)$, integrates in x , and then multiplies it by the right by $E_r(y)$, and integrates in y , it takes the form

$$\begin{aligned} & \int d^4x e^{iqx} \bar{E}_p(x) q^\mu \gamma_\mu E_r(x) \\ &= \int d^4x d^4y (e^{iqy} - e^{iqx}) \bar{E}_p(x) \bar{S}^{-1}(x, y) E_r(y). \end{aligned} \quad (15)$$

Using the properties of the E_p functions, the integration on the left-hand side (LHS) of the identity (15) can be done [7,8] to obtain

$$\begin{aligned} & \int d^4x e^{iqx} \bar{E}_p(x) q^\mu \gamma_\mu E_r(x) \\ &= (2\pi)^4 \delta^{(3)}(r + q - p) e^{iq_1(r_2 + p_2)/2eH} e^{-\hat{q}_\perp^2/2} \\ &\quad \times \sum_{\sigma_p \sigma_r} \frac{1}{\sqrt{n_p! n_r!}} e^{i \text{sgn}(eH)(n_p - n_r)\varphi} J_{n_p n_r}(\hat{q}_\perp) \Delta_p q^\mu \gamma_\mu \Delta_r. \end{aligned} \quad (16)$$

In Eq. (16) we used the following notation:

$$\begin{aligned} J_{n_p n_r}(\hat{q}_\perp) &\equiv \sum_{m=0}^{\min(n_p, n_r)} \frac{n_p! n_r!}{m! (n_p - m)! (n_r - m)!} \\ &\quad \times [i \text{sgn}(eH) \hat{q}_\perp]^{n_p + n_r - 2m}, \\ \delta^{(3)}(r + q - p) &\equiv \delta(r_0 + q_0 - p_0) \delta(r_2 + q_2 - p_2) \\ &\quad \times \delta(r_3 + q_3 - p_3), \\ \Delta_p &\equiv \Delta(\sigma_p), \quad n_p \equiv n(k_p, \sigma_p), \\ \Delta_r &\equiv \Delta(\sigma_r), \quad n_r \equiv n(k_r, \sigma_r), \end{aligned} \quad (17)$$

along with the dimensionless variables

$$\hat{q}_\mu \equiv \frac{q_\mu \sqrt{2|eH|}}{2eH}, \quad \mu = 0, 1, 2, 3, \quad (18)$$

and their corresponding polar coordinates

$$\hat{q}_\perp \equiv \sqrt{\hat{q}_1^2 + \hat{q}_2^2}, \quad \varphi \equiv \arctan(\hat{q}_2 / \hat{q}_1). \quad (19)$$

Introducing now the function

$$\begin{aligned} \Phi_{n_p n_r}(r, q, p) &\equiv (2\pi)^4 \delta^{(3)}(r + q - p) e^{iq_1(r_2 + p_2)/2eH} e^{-\hat{q}_\perp^2/2} \\ &\quad \times \frac{1}{\sqrt{n_p! n_r!}} e^{i \text{sgn}(eH)(n_p - n_r)\varphi} J_{n_p n_r}(\hat{q}_\perp), \end{aligned} \quad (20)$$

it is possible to express Eq. (16) in a more compact way:

$$\begin{aligned}
& \int d^4x e^{iqx} \bar{E}_p(x) q^\mu \gamma_\mu E_r(x) \\
&= \sum_{\sigma_p \sigma_r} \Phi_{n_p n_r}(r, q, p) \Delta_p \gamma_\mu q^\mu \Delta_r. \quad (21)
\end{aligned}$$

One can work in a similar way with the right-hand side of Eq. (15) in order to express it also in terms of the Φ function (20). With that aim, let us recall that the E_p transform [23] of the inverse fermion propagator satisfies

$$\bar{S}^{-1}(x, y) = \sum_{k_p} \int dp_0 dp_2 dp_3 E_p(x) [\gamma \cdot \bar{p} + \tilde{\Sigma}_A(\bar{p})] \bar{E}_p(y), \quad (22)$$

where $\tilde{\Sigma}_A(\bar{p})$ is the eigenvalue matrix of the mass operator introduced in Eq. (13).

Substituting Eqs. (21) and (22) back in Eq. (15), we obtain

$$\sum_{\sigma_p \sigma_r} \Phi_{n_p n_r}(r, q, p) \Delta_p \gamma_\mu q^\mu \Delta_r = \int d^4x d^4y (e^{iqy} - e^{iqx}) \bar{E}_p(x) \left\{ \sum_{k'} \int dp'_0 dp'_2 dp'_3 E_{p'}(x) [\gamma \cdot \bar{p}' + \tilde{\Sigma}_A(\bar{p}')] \bar{E}_{p'}(y) \right\} E_r(y). \quad (23)$$

There are two different types of integrals in space coordinates on the right-hand side of Eq. (23). One involves only the product of two E_p functions, and so it is solved using the orthonormality relation (10). The other integral is of the form

$$\begin{aligned}
& \int d^4x e^{iqx} \bar{E}_p(x) E_{p'}(x) = (2\pi)^4 \delta^{(3)}(p' + q - p) e^{iq_1(p'_2 + p_2)/2eH} e^{-\hat{q}_\perp^2/2} \sum_{\sigma_p \sigma_{p'}} \frac{1}{\sqrt{n_p! n_{p'}!}} e^{i \operatorname{sgn}(eH)(n_p - n_{p'})\varphi} J_{n_p n_{p'}}(\hat{q}_\perp) \Delta_p \Delta_{p'} \\
&= \sum_{\sigma_p \sigma_{p'}} \Phi_{n_p, n_{p'}}(p', q, p) \Delta_p \Delta_{p'}. \quad (24)
\end{aligned}$$

Integrating in x and y in Eq. (23) and using Eqs. (10) and (24), we obtain

$$\begin{aligned}
\sum_{\sigma_p \sigma_r} \Phi_{n_p n_r}(r, q, p) \Delta_p \gamma_\mu q^\mu \Delta_r &= \sum_{k'} \int dp'_0 dp'_2 dp'_3 \left\{ (2\pi)^4 \delta^{(4)}(p - p') [\gamma \cdot \bar{p}' + \tilde{\Sigma}_A(\bar{p}')] \sum_{\sigma_{p'} \sigma_r} \Phi_{n_p, n_{p'}}(r, q, p') \Delta_{p'} \Delta_r \right\} \\
&\quad - \sum_{k'} \int dp'_0 dp'_2 dp'_3 \left\{ (2\pi)^4 \delta^{(4)}(p' - r) \sum_{\sigma_p \sigma_{p'}} \Phi_{n_p n_{p'}}(p', q, p) \Delta_p \Delta_{p'} [\gamma \cdot \bar{p}' + \tilde{\Sigma}_A(\bar{p}')] \right\}. \quad (25)
\end{aligned}$$

Thanks to the delta functions, the evaluation of the integrals in p'_i and the sum in k' in the identity (25) is straightforward. Thus,

$$\sum_{\sigma_p \sigma_r} \Phi_{n_p n_r}(r, q, p) \Delta_p \gamma_\mu q^\mu \Delta_r = \sum_{\sigma_p \sigma_r} \Phi_{n_p n_r}(r, q, p) \delta_{\sigma_p \sigma_r} \{ [\gamma \cdot \bar{p} + \tilde{\Sigma}_A(\bar{p})] \Delta_p - \Delta_p [\gamma \cdot \bar{r} + \tilde{\Sigma}_A(\bar{r})] \}, \quad (26)$$

where we used

$$\Delta(\sigma) \Delta(\sigma') = \delta_{\sigma \sigma'} \Delta(\sigma). \quad (27)$$

The matrices $\Delta(\sigma)$ satisfy the commutation relations

$$\Delta \gamma_\parallel^\mu = \gamma_\parallel^\mu \Delta \quad \text{with} \quad \gamma_\parallel^\mu = (\gamma^0, 0, 0, \gamma^3), \quad (28)$$

$$\Delta \gamma_\perp^\mu = \gamma_\perp^\mu (1 - \Delta) \quad \text{with} \quad \gamma_\perp^\mu = (0, \gamma^1, \gamma^2, 0). \quad (29)$$

Therefore, the matrix structure on the LHS of Eq. (26) can be written as

$$\Delta_p \gamma_\mu q^\mu \Delta_r = (\gamma_\perp \cdot q_\perp) \Delta_r (1 - \delta_{\sigma_p \sigma_r}) + (\gamma_\parallel \cdot q_\parallel) \Delta_r \delta_{\sigma_p \sigma_r}. \quad (30)$$

Using the above equations, and recalling that the function $\Phi_{n_p n_r}$ contains a tridimensional delta function $\delta^{(3)}(r + q - p)$, one can eliminate the term proportional to $\gamma_\parallel \cdot q_\parallel$ with a similar term proportional to $\gamma_\parallel \cdot (p - r)_\parallel$ on the RHS of Eq. (26). Then, the final expression in momentum space of the WT identity in ladder QED with an external magnetic field becomes

$$\begin{aligned}
& \sum_{\sigma_p \sigma_r} \Phi_{n_p n_r}(r, q, p) \{ (\gamma_\perp \cdot q_\perp) \Delta_r (1 - \delta_{\sigma_p \sigma_r}) \\
&\quad - [\gamma_\perp \cdot (\bar{p}_\perp + \bar{r}_\perp) \Delta_p - \gamma_\perp \cdot \bar{r}_\perp] \delta_{\sigma_p \sigma_r} \\
&\quad - [\tilde{\Sigma}_A(\bar{p}) \Delta_p - \Delta_r \tilde{\Sigma}_A(\bar{r})] \delta_{\sigma_p \sigma_r} \} = 0. \quad (31)
\end{aligned}$$

It is evident that Eq. (31) is showing already some characteristic signs of the theory in the presence of a magnetic field. For instance, one can notice a separation between parallel- and perpendicular-to-the-field variables. It also contains extra structures (like those formed by the products of γ matrices times the matrices Δ_r), which are not present in the case without an external field. Unless the structure of the mass operator eigenvalue matrix $\tilde{\Sigma}_A$ is specified, we cannot find the solution of this WT identity. In the next section, we discuss the invariant structure of $\tilde{\Sigma}_A$ in the presence of a constant magnetic field.

III. MASS OPERATOR STRUCTURE

The structure of the fermion mass operator can be found on very general grounds. Since the mass operator must be a γ -matrix-based scalar function, it can be represented as a sum of Lorentz contractions between the tensorial structures that can be formed using the available vectors and tensors of the theory, and the elements of the Clifford algebra generated by the Dirac matrices and their products. The possible contractions are restricted, however, by the invariances of QED. That is, the mass operator must be invariant under Lorentz, C, P, and T transformations.

In QED, because the only tensorial structure available is the vector momentum p_μ , the above requirements lead to the familiar structure

$$\tilde{\Sigma}(p) = Z\gamma \cdot p + mI. \quad (32)$$

Here Z and m are functions of the magnitude of p . Their values depend on the approximation used to calculate $\tilde{\Sigma}$.

When a constant external electromagnetic field is present, in addition to the momentum p_μ , the theory contains the field strength $F_{\mu\nu}$. Then, in the presence of an electromagnetic field, the general structure of the mass operator becomes [22,24]

$$\tilde{\Sigma}_A(p, F) = i\gamma \cdot V + mI + \sigma^{\mu\nu} T_{\mu\nu} + i\gamma_5 \gamma^\mu A_\mu + R\gamma_5 F_{\mu\nu}^* F^{\mu\nu}, \quad (33)$$

where, besides the new term with the pseudoscalar $F_{\mu\nu}^* F^{\mu\nu}$, we have now contractions with a vector V_μ , a pseudovector A_μ , and a tensor $T_{\mu\nu}$, given, respectively, by

$$V_\mu = Zp_\mu + Z' e^2 F_{\mu\nu} F^{\nu\rho} p_\rho, \quad (34)$$

$$A_\mu = A e F_{\mu\nu}^* p^\nu, \quad (35)$$

$$T_{\mu\nu} = T e F_{\mu\nu} + T' e (p_\mu F_{\nu\rho} - p_\nu F_{\mu\rho}) p_\rho. \quad (36)$$

It can be checked that Eq. (33) satisfies all the required invariances of the mass operator in QED. It should be pointed out, however, that the second term in Eq. (36) was not considered in Ref. [22]. This term was introduced for the first time in Ref. [24], since the structure it generates when contracted with $\sigma^{\mu\nu}$ satisfies all the invariance requirements, and therefore, there is no reason to leave it out of the general

structure of $\tilde{\Sigma}_A$. Note that the structure $i\gamma \cdot (Fp)$ is absent from Eq. (33), since it violates charge conjugation:

$$C\gamma^0 \tilde{\Sigma}_A^*(p, F) \gamma^0 C^{-1} = \tilde{\Sigma}_A(p, -F). \quad (37)$$

In Eqs. (33)–(36) the coefficients Z , Z' , A , T , T' , m , and R are functions of the magnitudes of the momentum and the external field strength.

For a constant magnetic background field, the last term of Eq. (33) vanishes. If we assume that the magnetic field is along the three-axis, the only nonzero component of $F_{\mu\nu}$ is $F_{12} = H$. Then the mass operator (33) can be written as

$$\begin{aligned} \tilde{\Sigma}_A(p, F) = & iZ_{||} \gamma \cdot p_{||} + iZ_{\perp} \gamma \cdot p_{\perp} + mI + 2eHS\Sigma_3 \\ & + iAeH\Sigma_3 \gamma \cdot p_{||}, \end{aligned} \quad (38)$$

where we used the following definitions and equations:

$$Z_{||} \equiv Z, \quad (39)$$

$$Z_{\perp} \equiv Z - Z' e^2 H^2, \quad (40)$$

$$S \equiv (T - T' p_{\perp}^2), \quad (41)$$

$$\Sigma_3 = \sigma^{12} = i\gamma^1 \gamma^2, \quad (42)$$

$$\gamma^5 \gamma^0 = \Sigma_3 \gamma^3, \quad (43)$$

$$\gamma^5 \gamma^3 = \Sigma_3 \gamma^0. \quad (44)$$

It is timely to recall that our goal in this section is to find the structure of the eigenvalue matrix of the mass operator in momentum coordinates $\tilde{\Sigma}_A(\bar{p})$, as defined in Eq. (13); so we can use it later to solve the WT identity (31). From the discussion leading to Eq. (38), it is obvious that the structure of $\tilde{\Sigma}_A(\bar{p})$ will be given by Eq. (38) with p substituted by \bar{p} :

$$\begin{aligned} \tilde{\Sigma}_A(\bar{p}) = & iZ_{||}(\bar{p}, F) \gamma \cdot \bar{p}_{||} + iZ_{\perp}(\bar{p}, F) \gamma \cdot \bar{p}_{\perp} + m(\bar{p}, F)I \\ & + 2eHS(\bar{p}, F)\Sigma_3 + iA(\bar{p}, F)eH\Sigma_3 \gamma \cdot \bar{p}_{||}. \end{aligned} \quad (45)$$

Note the separation between parallel and perpendicular variables, typical of the theory in the presence of a magnetic field. Because of this separation, we have two Z coefficients, $Z_{||}$ and Z_{\perp} , compared to only one in the case with no magnetic field. Another physically expected new term is the one related to the spin-field interaction [the fourth term in Eq. (45)]. The interpretation of the new term with coefficient A is less transparent. However, as we show below, the ladder WT identity requires the A coefficient to be zero, although it allows a solution on which the spin-field interaction term is present.

IV. SOLUTIONS OF THE LADDER WT IDENTITY IN AN EXTERNAL MAGNETIC FIELD

To find the solution of the WT identity (31), we just need to substitute the mass operator (45) in Eq. (31), and then find

the values of the coefficients that solve the WT identity. As discussed in the Introduction, we are interested in determining if the solutions of the ladder SD equation for the fermion propagator [7,8] satisfy the WT identity when this one is taken in the same approximation.

With that aim, let us express the WT identity in a more convenient way, before making explicit use of Eq. (45). First, note that the function $\Phi_{n_p n_r}(r, q, p)$, which appears multiplying all the terms in the WT identity (31), contains an exponential factor $e^{-\hat{q}_\perp^2/2}$ [see the definition of $\Phi_{n_p n_r}$, Eq. (20)]. Notice also that the dimensionless variable \hat{q}_\perp^2 is related to the perpendicular momentum of the photon and is not linked to the arguments

$$\bar{p} = (p_0, 0, -\text{sgn}(eH)\sqrt{2|eH|k_p}, p_3)$$

and

$$\bar{r} = (r_0, 0, -\text{sgn}(eH)\sqrt{2|eH|k_r}, r_3)$$

of the mass operator in Eq. (31). For large \hat{q}_\perp the exponential function $e^{-\hat{q}_\perp^2/2}$ ensures that the identity is always approxi-

mately satisfied if the mass operator is finite in any region of the arguments \bar{p} and \bar{r} . This last assumption, although rather restrictive, is in agreement with the results of Hong, Kim, and Sin [10] for the behavior of the mass operator in the infrared and ultraviolet regions.

For small \hat{q}_\perp , one can approximate [7,8] the function $J_{n_p n_r}(\hat{q}_\perp)$ appearing in the definition of $\Phi_{n_p n_r}$, Eq. (20), as

$$J_{n_p n_r}(\hat{q}_\perp) \cong n_p! \delta_{n_p n_r} \quad (46)$$

to obtain

$$\begin{aligned} \Phi_{n_p n_r}(r, q, p) \\ \cong (2\pi)^4 \delta^{(3)}(r + q - p) e^{iq_1(r_2 + p_2)/2eH} e^{-\hat{q}_\perp^2/2} \delta_{n_p n_r}. \end{aligned} \quad (47)$$

Substituting Eq. (47) in the WT identity (31), it can be written as

$$\begin{aligned} \sum_{\sigma_p \sigma_r} \delta_{k_p + [\text{sgn}(eH)/2]\sigma_p, k_r + [\text{sgn}(eH)/2]\sigma_r} (\gamma_\perp \cdot q_\perp) \Delta_r \\ = \sum_{\sigma_p \sigma_r} \delta_{\sigma_p \sigma_r} \delta_{k_p + [\text{sgn}(eH)/2]\sigma_p, k_r + [\text{sgn}(eH)/2]\sigma_r} \{ (\gamma_\perp \cdot q_\perp) \Delta_r + \gamma_\perp \cdot (\bar{p}_\perp + \bar{r}_\perp) \Delta_p - (\gamma_\perp \cdot \bar{r}_\perp) + \tilde{\Sigma}_A(\bar{p}) \Delta_p - \Delta_r \tilde{\Sigma}_A(\bar{r}) \}, \end{aligned} \quad (48)$$

where we cancelled out the common functions in both sides. If we sum in σ_r on the RHS of Eq. (48), and rename σ_p as σ , we find

$$\begin{aligned} \sum_{\sigma_p \sigma_r} \delta_{k_p + [\text{sgn}(eH)/2]\sigma_p, k_r + [\text{sgn}(eH)/2]\sigma_r} (\gamma_\perp \cdot q_\perp) \Delta_r \\ = \sum_{\sigma} \delta_{k_p k_r} \{ (\gamma_\perp \cdot q_\perp) \Delta(\sigma) + \gamma_\perp \cdot (\bar{p}_\perp + \bar{r}_\perp) \Delta(\sigma) - (\gamma_\perp \cdot \bar{r}_\perp) + \tilde{\Sigma}_A(\bar{p}) \Delta(\sigma) - \Delta(\sigma) \tilde{\Sigma}_A(\bar{r}) \}. \end{aligned} \quad (49)$$

Summing now in all σ variables, and using the relation $\Delta(1) + \Delta(-1) = 1$, it is straightforwardly found that the WT identity reduces to

$$\begin{aligned} (\gamma_\perp \cdot q_\perp) [\delta_{k_p, k_r + \text{sgn}(eH)} \Delta(1) + \delta_{k_p, k_r - \text{sgn}(eH)} \Delta(-1)] \\ = \delta_{k_p k_r} \{ iZ_{||}(\bar{p}, F) \gamma \cdot \bar{p}_{||} - iZ_{||}(\bar{r}, F) \gamma \cdot \bar{r}_{||} + i[Z_\perp(\bar{p}, F) - Z_\perp(\bar{r}, F)] \gamma \cdot \bar{p}_\perp + [m(\bar{p}, F) - m(\bar{r}, F)] I \\ + 2eH[S(\bar{p}, F) - S(\bar{r}, F)] \Sigma_3 + iA(\bar{p}, F) eH \Sigma_3 \gamma \cdot \bar{p}_{||} - iA(\bar{r}, F) eH \Sigma_3 \gamma \cdot \bar{r}_{||} \}. \end{aligned} \quad (50)$$

In Eq. (50) we already substituted the mass operators appearing in Eq. (49) by their explicit expressions according to Eq. (45), and used that $\bar{p}_\perp = \bar{r}_\perp$ in all terms multiplied by $\delta_{k_p k_r}$.

The origin of the term on the LHS of Eq. (50) can be traced to the E_p transform of the vertex. We might think that

because of this term there are no self-consistent solutions of the ladder WT identity. Fortunately, we have not yet considered all the restrictions of the approximation used in the calculation of the ladder SD solutions [7,8]. To work in the same approximation, we need to constrain the fermions to the LLL. This means that we must evaluate Eq. (50) in $k_p = k_r = 0$.

Therefore, for fermions in the LLL, the WT identity is simply equivalent to

$$\begin{aligned}
 & iZ_{||}(\bar{p}, F) \gamma \cdot \bar{p}_{||} + m(\bar{p}, F) I + 2eHS(\bar{p}, F) \Sigma_3 \\
 & + iA(\bar{p}, F) eH \Sigma_3 \gamma \cdot \bar{p}_{||} \\
 & = iZ_{||}(\bar{r}, F) \gamma \cdot \bar{r}_{||} + m(\bar{r}, F) I + 2eHS(\bar{r}, F) \Sigma_3 \\
 & + iA(\bar{r}, F) eH \Sigma_3 \gamma \cdot \bar{r}_{||}.
 \end{aligned} \tag{51}$$

Note that in this approximation Z_{\perp} disappears from the WT identity. The identity (51) is satisfied if

$$Z_{||}(\bar{p}, F) = Z_{||}(\bar{r}, F) = 0, \tag{52}$$

$$m(\bar{p}, F) = m(\bar{r}, F), \tag{53}$$

$$S(\bar{p}, F) = S(\bar{r}, F), \tag{54}$$

$$A(\bar{p}, F) = A(\bar{r}, F) = 0. \tag{55}$$

Comparing the solutions Eqs. (52)–(55) for the WT identity with the solutions found by Leung, Ng, and Arkley [7] and by Lee, Leung, and Ng [8], for the SD equation in the Feynman gauge using the same approximation, we see that they are consistent with each other. The SD solutions of these papers [7,8] can be summarized by

$$Z = Z_{||} = Z_{\perp} = 0, \tag{56}$$

$$m(\bar{p}, F) = m(\bar{r}, F) \approx m. \tag{57}$$

The other coefficients were assumed to be zero:

$$S(\bar{p}, F) = S(\bar{r}, F) = 0, \tag{58}$$

$$A(\bar{p}, F) = A(\bar{r}, F) = 0, \tag{59}$$

which, although not the most general solution, is, however, compatible with Eqs. (54) and (55).

We conclude in this way that the chiral symmetry breaking, found in ladder QED through the SD approach in the LLL limit in the Feynman gauge, is consistent with the corresponding WT identity. Therefore, the chiral symmetry breaking obtained by Gusynin, Miransky, and Shovkovy [6], and by Leung and co-workers [7,8] is not an artifact of the approximation used, but a gauge-independent result.

Magnetic-field-induced chiral symmetry breaking has been also found in non-Abelian models. To prove the gauge

invariance of the magnetic-field-induced chiral mass in a non-Abelian case, one would need to do a similar study on the consistency of the SD solution with the solution of the correspondent WT identity. However, the WT identity in this case must be substituted by the Slavnov-Taylor identity for the fermion-antifermion-gluon vertex taken in the same non-perturbative approximation on which the SD equation is solved. The main difficulty in this case is that the Slavnov-Taylor identity has not been extended beyond a perturbative analysis.

V. CONCLUSIONS

In this paper we have studied the WT identity for the fermion-antifermion-gauge boson vertex in massless QED in the presence of a constant magnetic field. The fermion Green's functions were considered exact in the external field, while the quantum effects were calculated within the ladder approximation, i.e., taking the bare vertex and the free photon propagator, but the full electron propagator. To find the WT identity in p -space, we worked in the E_p representation, where the fermion mass operator is diagonal in momentum coordinates.

Our approach led us to a nonperturbative WT identity involving the electron mass propagator in the presence of a magnetic field. To solve the identity we used the structure of the mass propagator in p -space in the presence of a magnetic field. The mass operator in momentum space was found by considering all the independent γ -matrix-based scalars that could be formed out of p , $F_{\mu\nu}$, and the elements of the Dirac Clifford algebra, which satisfy the symmetries of the theory.

For lowest-Landau-level fermions, we found an explicit solution of the ladder WT identity. The LLL case is specially important, since according to recent results in the problem of MCSB [6,8] it is precisely the LLL contribution that determines the appearance of a fermion mass in ladder QED with constant magnetic field. Since the solution of the SD equation that led to the existence of a chiral mass is in agreement with our general solution for the WT identity within the same approximation, we have effectively established that the dynamical chiral symmetry breaking due to a magnetic field is a gauge-invariant result.

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